(Hf) 1 a) $\displaystyle\int_{\ln 4}^{\ln 8} \frac{e^x}{e^{2x}-4}\,dx = (*)$

$t = e^x$. $\ln 4 \le x \le \ln 8 \Rightarrow 4 = e^{\ln 4} \le t = e^x \le e^{\ln 8} = 8$

$x = \ln(t) = g(t) \quad (4 \le t \le 8)$

$g'(t) = \frac{1}{t} > 0 \Rightarrow g\uparrow$ és $R_g = [\ln 4, \ln 8]$

$$(*) = \int_4^8 \frac{t}{t^2-4} \cdot \frac{1}{t}\,dt = \int_4^8 \frac{1}{t^2-4}\,dt = (**)$$

$$\frac{1}{t^2-4} = \frac{1}{(t-2)(t+2)} = \frac{A}{t-2} + \frac{B}{t+2} = \frac{A(t+2)+B(t-2)}{(t-2)(t+2)}$$

$\Rightarrow 1 = A(t+2) + B(t-2) \quad (t \in \mathbb{R})$

Ha $t=-2 \Rightarrow 1 = A\cdot 0 + B(-4) \Rightarrow B = -1/4$

Ha $t=2 \Rightarrow 1 = A\cdot 4 + B\cdot 0 \Rightarrow A = 1/4$

$\Rightarrow \frac{1}{t^2-4} = \frac{1/4}{t-2} - \frac{1/4}{t+2}$

$$(**) = \int_4^8 \left(\frac{1}{4}\frac{1}{t-2} - \frac{1}{4}\frac{1}{t+2}\right)dt = \left[\frac{1}{4}\ln(t-2) - \frac{1}{4}\ln(t+2)\right]_4^8 =$$

$$= \frac{1}{4}\Big[(\ln 6 - \ln 10) - (\ln 2 - \ln 6)\Big] = \frac{1}{4}(2\ln 6 - \ln 10 - \ln 2) =$$

$$= \frac{1}{4}\ln\frac{36}{20} = \frac{1}{4}\ln\frac{9}{5}.$$

Hf 1 b) $\int \frac{\sqrt{3x-1}}{x}\,dx = (*)$

$(x > 1/3)$

$t = \sqrt{3x-1} > 0 \Rightarrow t^2 = 3x-1 \Rightarrow x = \frac{t^2+1}{3} = g(t) \quad (t > 0)$

$g'(t) = \frac{2t}{3} > 0 \Rightarrow g \uparrow$ és $R_g = \left(\frac{1}{3}, +\infty\right)$

$$(*) = \int \frac{t}{\frac{t^2+1}{3}} \cdot \frac{2t}{3}\,dt = 2\int \frac{t^2}{t^2+1}\,dt = 2\int \frac{(t^2+1)-1}{t^2+1}\,dt =$$

$$= 2\int\left(1 - \frac{1}{t^2+1}\right)dt = 2\left(t - \operatorname{arctg} t\right) + c\,\Big|_{t=\sqrt{3x-1}} =$$

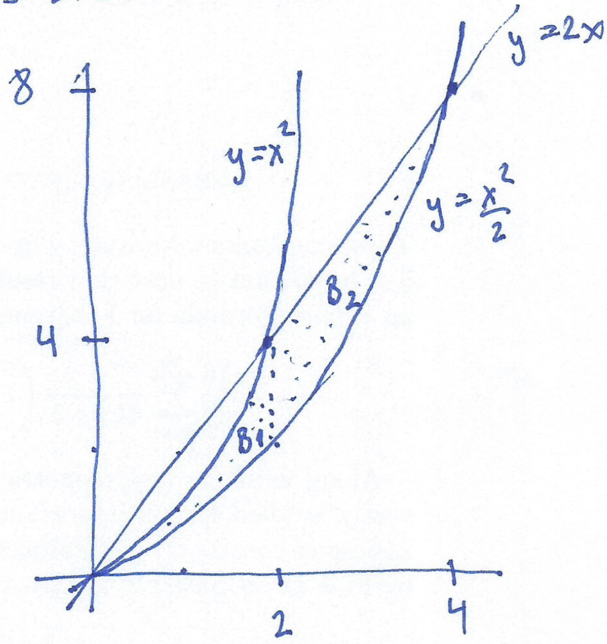
$$= 2\left(\sqrt{3x-1} - \operatorname{arctg}\sqrt{3x-1}\right) + C$$

---

1 c) $$\int_0^{+\infty} e^{-4x}\,dx = \lim_{t\to+\infty} \int_0^t e^{-4x}\,dx = \lim_{t\to+\infty}\left[\frac{e^{-4x}}{-4}\right]_0^t =$$

$$= \lim_{t\to+\infty}\left(-\frac{e^{-4t}}{4} - \left(-\frac{e^0}{4}\right)\right) = \lim_{t\to+\infty}\left(-\underbrace{\frac{1}{4e^{4t}}}_{\to 0} + \frac{1}{4}\right) = \frac{1}{4}$$

(Hf) 2 $y=x^2$, $y=\frac{x^3}{2}$ és $y=2x$
görbék által közrezárt korlátos síkidom területe!



$x^2 = 2x \quad \Rightarrow \underline{x=0}, \ \underline{x=2}$

$\frac{x^2}{2} = 2x \quad \Rightarrow x=0, \ \underline{x=4}$

$B_1 = \left\{ (x,y) \in \mathbb{R}^2 : 0 \le x \le 2, \ \frac{x^2}{2} \le y \le x^2 \right\}$

$B_2 = \left\{ (x,y) \in \mathbb{R}^2 : 2 \le x \le 4, \ \frac{x^2}{2} \le y \le 2x \right\}$

$$T(B_1) = \int_0^2 x^2 - \frac{x^2}{2}\, dx = \int_0^2 \frac{x^2}{2}\, dx =$$

$$= \left[\frac{x^3}{6} \right]_0^2 = \frac{2^3}{6} - 0 = \frac{4}{3}$$

$$T(B_2) = \int_2^4 2x - \frac{x^2}{2}\, dx = \left[x^2 - \frac{x^3}{6} \right]_2^4 = \left(16 - \frac{64}{6}\right) - \left(4 - \frac{8}{6}\right) = \frac{8}{3}$$

A keresett terület a két terület összege: $T = \frac{4}{3} + \frac{8}{3} = \underline{\underline{4}}$

---

3. Forgástest térfogata: $f(x) = \sqrt{\operatorname{arctg} x} \quad (0 \le x \le 1)$

$$V = \pi \int_a^b f^2(x)\, dx = \pi \int_0^1 \operatorname{arctg} x\, dx = \pi \left[x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) \right]_0^1 =$$

$$= \pi \left[\left(\operatorname{arctg} 1 - \frac{1}{2} \ln 2 \right) - \left(0 - \frac{1}{2} \ln 1 \right) \right] = \pi \left(\frac{\pi}{4} - \frac{1}{2} \ln 2 \right)$$

. . . .

$(x \in \mathbb{R})$ $\int \operatorname{arctg} x\, dx = \int 1 \cdot \operatorname{arctg} x\, dx = \int (x)' \operatorname{arctg} x\, dx =$

$$= x \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2}\, dx = x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2}\, dx =$$

$$= x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + C$$

Hf 4. Ívhossz: $f(x) = x^{3/2} \quad (0 \le x \le 4)$

$$f'(x) = \frac{3}{2} x^{1/2} \Rightarrow (f'(x))^2 = \frac{9}{4} x$$

$$l = \int_a^b \sqrt{1+(f'(x))^2}\, dx = \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx = \int_0^4 \sqrt{\frac{4+9x}{4}}\, dx =$$

$$= \frac{1}{2} \int_0^4 (4+9x)^{\frac{1}{2}}\, dx = \frac{1}{2} \left[\frac{(4+9x)^{3/2}}{3/2 \cdot 9} \right]_0^4 =$$

$$= \frac{1}{27} \left[\sqrt{(4+9x)^3} \right]_0^4 = \frac{1}{27} \left(\sqrt{40^3} - \sqrt{4^3} \right) =$$

$$= \frac{80}{27}\sqrt{10} - \frac{8}{27}$$