Hf

1. $f(x)=\sqrt{1+2x} \qquad \left(x>-\frac{1}{2}\right)$

a) $T_{2,0}f(x)=?$

$f(x)=\sqrt{1+2x}=(1+2x)^{\frac{1}{2}} \quad \to f(0)=1 \qquad /1$

$f'(x)=\frac{1}{2}(1+2x)^{-\frac{1}{2}}\cdot 2=(1+2x)^{-\frac{1}{2}} \quad \to f'(0)=1 \qquad /1$

$f''(x)=-\frac{1}{2}(1+2x)^{-\frac{3}{2}}\cdot 2=-(1+2x)^{-\frac{3}{2}} \quad \to f''(0)=-1 \qquad /2$

$$T_{2,0}f(x)=1+x-\frac{x^2}{2} \qquad (x\in\mathbb{R})$$

b) Hibabecslés a $\left[-\frac{5}{18},\frac{1}{4}\right]$ intervallumon.

- Legyen $x\in\left(0,\frac{1}{4}\right]$. Ekkor $\exists\, \xi_x\in(0,x)$, hogy

$$(*) \qquad f(x)-T_{2,0}f(x)=\frac{f'''(\xi_x)}{3!}x^3$$

$$f'''(x)=\frac{3}{2}(1+2x)^{-\frac{5}{2}}\cdot 2=\frac{3}{\sqrt{(1+2x)^5}}$$

Ezért

$$|f(x)-T_{2,0}f(x)|\le\frac{1}{6}\cdot\frac{3}{\sqrt{(1+2\xi_x)^5}}\cdot|x|^3<\frac{1}{6}\cdot\frac{3}{\sqrt{(1+0)^5}}\cdot\left(\frac{1}{4}\right)^3=\frac{1}{128}=0{,}0078125$$

- Legyen $x\in\left[-\frac{5}{18},0\right)$. Ekkor $\exists\,\xi_x\in(x,0)$, hogy $(*)$ teljesül.

Ezért $\left(\xi_x\in\left(-\frac{5}{18},0\right)\right)$

$$|f(x)-T_{2,0}f(x)|\le\frac{1}{6}\cdot\frac{3}{\sqrt{(1+2\xi_x)^5}}|x|^3\le\frac{1}{6}\cdot\frac{3}{\sqrt{\left(1-2\cdot\frac{5}{18}\right)^5}}\cdot\left|-\frac{5}{18}\right|^3=$$

$$=\frac{1}{6}\cdot\frac{3}{\sqrt{(4/9)^5}}\cdot\left(\frac{5}{18}\right)^3=\frac{3}{6}\cdot\left(\frac{3}{2}\right)^5\cdot\left(\frac{5}{18}\right)^3=\frac{125}{1536}=\underline{\underline{0{,}08138}}$$

2, Taylor sor

a) $f(x) = 2^x \quad (x \in \mathbb{R}) \quad a = 1.$

Tudjuk, hogy $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \quad (x \in \mathbb{R})$

Ekkor

$$2^x = e^{\ln 2^x} = e^{x \ln 2} = \sum_{n=0}^{+\infty} \frac{(x \ln 2)^n}{n!} = \sum_{n=0}^{+\infty} \frac{\ln^n 2}{n!} x^n \quad (x \in \mathbb{R})$$

Ezért

$$\underbrace{2^{x-1}}_{= \frac{2^x}{2}} = \sum_{n=0}^{+\infty} \frac{\ln^n 2}{n!} (x-1)^n \quad (x \in \mathbb{R})$$

Ezért $\underline{2^x} = 2 \sum_{n=0}^{+\infty} \frac{\ln^n 2}{n!} (x-1)^n = \underline{\sum_{n=0}^{+\infty} \frac{2 \cdot \ln^n 2}{n!} (x-1)^n \quad (x \in \mathbb{R})}$

b) $f(x) = \ln(x^2+1) \quad (x \in \mathbb{R}) \quad a = 0$

Előadáson igazoltuk, hogy

$$\ln(x+1) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \cdot x^n \quad (x \in (-1,1))$$

Ezért $\ln(x^2+1) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^{2n} \quad \underbrace{(x^2 \in (-1,1))}_{x \in (-1,1)}$