Hf

1 a) $\int \frac{x^3+x^2-x+3}{x^2+x-2}\,dx = \int \frac{x^3+x^2-2x+x+3}{x^2+x-2}\,dx =$

$(x>1)$

$= \int \frac{x(x^2+x-2)+x+3}{x^2+x-2}\,dx = \int\left(x+\frac{x+3}{x^2+x-2}\right)dx = (*)$

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$\frac{x+3}{x^2+x-2} = \frac{x+3}{(x-1)(x+2)} = \frac{A}{x-1}+\frac{B}{x+2} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$

$\Rightarrow \quad x+3 = A(x+2)+B(x-1) \quad (x\in\mathbb{R})$

Ha $x=1 \Rightarrow \quad 4 = A\cdot 3 + B\cdot 0$, azaz $A = 3/4$

Ha $x=-2 \Rightarrow \quad 1 = A\cdot 0 + B(-3)$, azaz $B = -1/3$.

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$(*) = \int\left(x+\frac{3}{4}\frac{1}{x-1}-\frac{1}{3}\frac{1}{x+2}\right)dx = \frac{x^2}{2}+\frac{3}{4}\ln|x-1|-\frac{1}{3}\ln|x+2|+C =$

$= \frac{x^2}{2}+\frac{3}{4}\ln(x-1)-\frac{1}{3}\ln(x+2)+C$

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1 b) $\int \frac{x^4-x^2+1}{x^2(x+1)}\,dx = \int \frac{x^4-x^2+1}{x^3+x^2}\,dx = \int\left(x-1+\frac{1}{x^2(x+1)}\right)dx =$

$(0<x<1)$

$x^4+0x^3-x^2+0x+1 : x^3+x^2+0x+0 = x-1$

$-\ \ x^4+x^3$

$\quad -x^3-x^2$

$-\ \ -x^3-x^2$

$\quad 0 \quad +1$

$= \int\left(x-1-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx =$

$= \frac{x^2}{2}-x-\ln x-\frac{1}{x}+\ln(x+1)+C$

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$\frac{1}{x^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1} = \frac{A x(x+1)+B(x+1)+Cx^2}{x^2(x+1)}$

Ekkor $\quad 1 = Ax(x+1)+B(x+1)+Cx^2$

Ha $x=0 \Rightarrow 1 = A\cdot 0+B\cdot 1+C\cdot 0$, azaz $B=1$

Ha $x=-1 \Rightarrow 1 = A\cdot 0+B\cdot 0+C\cdot 1$, azaz $C=1$

Ha $x=1 \Rightarrow 1 = A\cdot 2+B\cdot 2+C\cdot 1$, azaz $1 = 2A+2\cdot 1+1\cdot 1 \Rightarrow A=-1$

1 c) $\displaystyle\int \frac{x+1}{x^2+3x+4}\,dx = (*)$

$(x\in\mathbb{R})$ Vegyük észre, hogy $x^2+3x+4>0$ $(x\in\mathbb{R})$!

$$x+1 = \gamma(2x+3)+\delta \qquad (x\in\mathbb{R})$$

Ha $x=-3/2 \Rightarrow -\frac{3}{2}+1 = \gamma\cdot 0+\delta \Rightarrow \delta = -\frac{1}{2}$

Ha $x=0 \Rightarrow 0+1 = \gamma\cdot 3+\delta \Rightarrow 1 = 3\gamma-\frac{1}{2} \Rightarrow \gamma = \frac{1}{2}$

$$(*) = \int \frac{1}{2}\cdot\frac{2x+3}{x^2+3x+4}\,dx - \int \frac{1}{2}\,\frac{1}{x^2+3x+4}\,dx = (**)$$

$$\int \frac{1}{x^2+3x+4}\,dx = \int \frac{1}{\left(x+\frac{3}{2}\right)^2+4-\frac{9}{4}}\,dx = \int \frac{1}{\left(x+\frac{3}{2}\right)^2+\frac{7}{4}}\,dx =$$

$$= \frac{4}{7}\int \frac{1}{\frac{4}{7}\left(x+\frac{3}{2}\right)^2+1}\,dx = \frac{4}{7}\int \frac{1}{\left(\frac{2x+3}{\sqrt{7}}\right)^2+1} = \frac{4}{7}\,\frac{\operatorname{arctg}\left(\frac{2x+3}{\sqrt{7}}\right)}{2/\sqrt{7}} + C$$

$$(**) = \frac{1}{2}\ln(x^2+3x+4) - \frac{1}{2}\cdot\frac{4}{7}\cdot\frac{\sqrt{7}}{2}\operatorname{arctg}\left(\frac{2x+3}{\sqrt{7}}\right)+C =$$

$$= \frac{1}{2}\ln(x^2+3x+4) - \frac{1}{\sqrt{7}}\operatorname{arctg}\left(\frac{2x+3}{\sqrt{7}}\right)+C$$

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1 d) $\displaystyle\int \frac{2x^2+x+1}{x^2(x^2+1)}\,dx = (*)$

$(x>0)$

$$\frac{2x^2+x+1}{x^2(x^2+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1} = \frac{Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2}{x^2(x^2+1)} \qquad (x\in\mathbb{R})$$

Ekkor $2x^2+x+1 = Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2$

Ha $x=0 \Rightarrow 1 = A\cdot 0+B\cdot 1+0 \Rightarrow B=1$

Ha $x=1 \Rightarrow 4 = A\cdot 2+B\cdot 2+C+D \Rightarrow 2A+C+D=2$

Ha $x=-1 \Rightarrow 2 = A(-2)+B\cdot 2-C+D \Rightarrow -2A-C+D=0$

összeadva $2D=2 \Rightarrow D=1$ és $2A+C=1$

Ha $x=2 \Rightarrow 11 = A\cdot 10+\underbrace{B\cdot 5}_{5}+(2C+\underbrace{D}_{1})\cdot 4 \Rightarrow 10A+8C=2 \Rightarrow 5A+4C=1$

$\Rightarrow 5A+4(1-2A)=1$

$\Rightarrow -3A=-3 \Rightarrow A=1 \Rightarrow C=-1$

$$(*) = \int\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1-x}{x^2+1}\right)dx =$$

$$= \int\left(\frac{1}{x}+\frac{1}{x^2}-\frac{1}{2}\,\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)dx = \ln x-\frac{1}{x}-\frac{1}{2}\ln(x^2+1)+\operatorname{arctg}x+C$$